The Mathematical Crusade '10

Four Questions

- There are 12 questions in total, each worth 10 points.
- At any given point of time, you will not have more than 4 questions to attempt.
- You are not expected to solve every question.
- If you manage to solve a question, submit the answer immediately.
- If you believe you cannot solve a question, you are welcome to pass it. You may not attempt a passed question again.
- Once you attempt or pass a question you will be given the next question, until the event ends or we run out of questions.
- All questions are problem solving questions. You are required to write complete solutions with proof.
- Question 1 is the Clueless clue in the Four Questions event.
- Ensure that your school name and school registration number is clearly written and marked on every sheet you submit.
- If you have any further queries, do not hesitate to ask the student volunteer or the teacher in-charge in your classroom.

Q1. Goldbach's strong conjecture states that every even integer greater than 4 can be expressed as the sum of 2 primes, not necessarily distinct. Goldbach's weak conjecture states that every odd number greater than 7 can be expressed as the sum of 3 primes. Show that Goldbach's weak conjecture follows from his strong conjecture!?

Q2. S(k)= 1! + 2! + 3! + ... + k!
For what values of k is S(k) a perfect square?

Q3. Prove that a plane can be perfectly tiled using an infinite number of congruent regular triangles, squares and hexagons. Further prove that it cannot be tiled by any other regular polygon.

Q4. A play has the following rules:

- There are fruits kept at integral points on the coordinate plane bounded by the positive x and y axis. There is one poisonous fruit at (0,0).
- If a player eats a fruit kept at (x,y) , all fruits at points (a,b) where a>=x, and b>=y are damaged, i.e. the second player cannot choose them anymore.
- The game finishes when one of the players dies, i.e., eats the poisonous fruit.

Prove that the both players cannot survive forever.

Q5. Let m and n be natural numbers. Given that mn + 1 is divisible by 24, show that m+n is divisible by 24.

Q6. Prove that there exist two powers of 2 which differ by a multiple of 1987.

Q7. Nine numbers are placed around a circle: four 1s and five 0s. The following operation is performed on the numbers: between each adjacent pair of numbers is placed a 0 if the numbers are different and a 1 if the numbers are the same. The 'old' numbers are then erased. After several of these operations, can all the remaining numbers be equal?

Q8. Show that the following inequality holds for all positive real numbers a, b, and c:

$$(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{2}$$

Q9. If you choose any n+1 numbers from the set {1,2,...,2n} prove that, there will always be two numbers such that one divides the other.

Q10. Show that the number 10^{10} cannot be written as the product of two natural numbers which do not contain the digit 0 in their decimal representation.

Q11. How many diagonals are there in an n-sided convex polygon?

Q12. On an 8x8 chessboard, coins are placed alternately heads and tails, with heads on black squares and tails on white squares. In a given step, any two coins are chosen and flipped. Can there, after any number of steps, remain only 1 head on the chessboard?