The Mathematical Crusade '10

Four Questions

- There are 12 questions in total, each worth 10 points.
- At any given point of time, you will not have more than 4 questions to attempt.
- If you manage to solve a question, submit the answer immediately.
- If you believe you cannot solve a question, you are welcome to pass it.
- Once you attempt or pass a question you will be given the next question, until the event ends or we run out of questions.
- All questions are problem solving questions. You are required to write complete solutions with proof.
- Ensure that your school name and school registration number is clearly written and marked on every sheet you submit.
- If you have any further queries, do not hesitate to ask the student volunteer or the teacher in-charge in your classroom.

Q1. Nine numbers are placed around a circle: four 1s and five 0s. The following operation is performed on the numbers: between each adjacent pair of numbers is placed a 0 if the numbers are different and a 1 if the numbers are the same. The 'old' numbers are then erased. After several of these operations, can all the remaining numbers be equal?

Q2. $S(k) = 1! + 2! + 3! + ... + k!$
For what values of k is S(k) a perfect square?

Q3. Prove that a plane can be perfectly tiled using an infinite number of congruent regular triangles, squares and hexagons. Further prove that it cannot be tiled by any other regular polygon.

Q4. A play has the following rules:

- There are fruits kept at integral points on the coordinate plane bounded by the positive x and y axis. There is one poisonous fruit at (0,0).
- If a player eats a fruit kept at (x,y) , all fruits at points (a,b) where a>=x, and b>=y are damaged, i.e. the second player cannot choose them anymore.
- The game finishes when one of the players dies, i.e., eats the poisonous fruit.

Prove that the both players cannot survive forever.

Q5. Let m and n be natural numbers. Given that mn + 1 is divisible by 24, show that m+n is divisible by 24.

Q6. Prove that there exist two powers of 2 which differ by a multiple of 1987.

Q7. Goldbach's strong conjecture states that every even integer greater than 4 can be expressed as the sum of 2 primes, not necessarily distinct. Goldbach's weak conjecture states that every odd number greater than 7 can be expressed as the sum of 3 primes. Show that Goldbach's weak conjecture follows from his strong conjecture!?

Q8. Show that the following inequality holds for all positive real numbers a, b, and c:

$$(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{2}$$

Q9. If you choose any n+1 numbers from the set {1,2,...,2n} prove that, there will always be two numbers such that one divides the other.

Q10. Show that the number 10^{10} cannot be written as the product of two natural numbers which do not contain the digit 0 in their decimal representation.

Q11. How many diagonals are there in an n-sided convex polygon?

Q12. On an 8x8 chessboard, coins are placed alternately heads and tails, with heads on black squares and tails on white squares. In a given step, any two coins are chosen and flipped. Can there, after any number of steps, remain only 1 head on the chessboard?